Implementation of the lake parameterisation scheme FLake into the numerical weather prediction model COSMO

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The application of the lake model FLake to represent the effect of lakes in numerical weather prediction (NWP) and climate models is discussed. As a lake parameterisation scheme FLake is implemented into the limited-area NWP model COSMO. Results from a numerical experiment with the coupled COSMO-FLake system, including the complete COSMO-model data assimilation cycle, indicate a good performance with respect to the lake surface temperature and to the freeze-up of lakes and the ice break-up. The use of FLake removes a significant overestimation of the lake surface temperature during winter that is typical of the routine COSMO sea surface temperature analysis. Some challenging issues are discussed, such as the development of external-parameter fields and the lake temperature spin-up following a cold start of FLake in an NWP or climate model.

Introduction

Lakes significantly affect the structure of the atmospheric boundary layer and therefore the surface fluxes of heat, water vapour and momentum. In most numerical weather prediction (NWP) and climate models, the effect of lakes is either entirely ignored or is parameterised very crudely. In coarse resolution models, a large number of small-to-medium size lakes are indistinguishable sub-grid scale features. These lakes become resolved scale features as the horizontal resolution is increased. Then, a physically sound model (parameterisation scheme) is required to predict the lake surface temperature and the effect of lakes on the structure and transport properties of the atmospheric boundary layer. Apart from being physically sound, a lake model must meet stringent requirements of computational economy. The terms “model” and “parameterisation scheme” may be used interchangeably in this context. The term “parameterisation scheme” is commonly used in the NWP and climate modelling community to discriminate a component (module) of a complex modelling system from its host that is referred to as an NWP (climate) model.

There are several aspects of the problem. For one thing, the interaction of the atmosphere
with the underlying surface is strongly dependent on the surface temperature. It is common for NWP models to assume that the water surface temperature can be kept constant over the forecast period. The assumption is to some extent justified for seas and deep lakes. It is doubtful for small-to-medium size relatively shallow lakes, where diurnal variations of the surface temperature reach several degrees. A large number of such lakes become resolved scale features as the horizontal resolution is increased. The use of a horizontal grid size of about three kilometres or even less has already become a common practice in short-range weather forecast. In NWP models with coarser resolution, many small-to-medium size lakes remain sub-grid scale features. However, the presence of these lakes cannot be ignored due to their aggregate effect on the grid-scale surface fluxes. This also holds for climate models concerned with the time scales ranging from many days to many years.

Initialisation of the water-type grid points of an NWP model presents considerable difficulties. When the observational data for several water points are not available, these points are initialised by means of interpolation between the nearest water-type points for which the surface temperature is known (some NWP models also account for the two-metre temperature observations over land in the neighbourhood of the water points in question). Such procedure is not too inaccurate for sea points. Large horizontal gradients of the sea surface temperature (SST) are comparatively rare, so that the interpolated SST is expected to be a reasonably good approximation of the actual SST. In contrast to open sea, lakes are enclosed water bodies of a comparatively small horizontal extent. The lake surface temperature has little or nothing to do with the surface temperature obtained by means of interpolation between the alien water bodies.

Another important aspect of the problem is that lakes strongly modify the structure and the transport properties of the atmospheric boundary layer. One of the outstanding questions is the parameterisation of the roughness of the water surface with respect to wind and to scalar quantities (e.g. the effect of limited wind fetch on the momentum transfer should be accounted for). A consideration of this aspect of the lake parameterisation problem is beyond the scope of the present paper.

An interest in the problem of lakes has led to the development of several lake models for use in NWP and climate studies. Three-dimensional lake models account for both vertical and horizontal transport of momentum and heat and provide detailed information about the lake temperature structure (Song et al. 2004, León et al. 2005, 2007, Long et al. 2007). However, a high computational cost limits their utility to only a few large lakes and to research applications. The use of three-dimensional lake models (or ocean models customised for lakes) as lake parameterisation schemes in NWP and other operational applications will most likely be impossible for some years to come. One-dimensional lake models range from the simplest one-layer slab models to rather sophisticated turbulence closure models. One-layer models characterise the entire water column by a single value of temperature, assuming a complete mixing down to the lake bottom (Ljungemyr et al. 1996), or to the bottom of a mixed layer of a fixed depth which may vary spatially (Goyette et al. 2000). Although this assumption results in a bulk model that is computationally very efficient, it is an oversimplification from the physical point of view. As most lakes are stratified over a considerable part of the year, neglecting the lake thermocline results in large errors in the surface temperature. Second-order turbulence closure models, e.g. models that carry transport equations for the turbulence kinetic energy (TKE) and its dissipation rate (Omstedt 1999, Blencker et al. 2002, Stepanenko 2005, Stepanenko et al. 2006) or for the TKE only (Tsuang et al. 2001), may describe the lake thermocline with reasonable accuracy. These models are computationally rather expensive, however. Their use to treat a large number of lakes can hardly be afforded in operational applications. Hostetler (see Hostetler and Bartlein 1990, Hostetler 1991, Hostetler et al. 1993, Barrette and Laprise 2005) developed a lake model that uses an algebraic stability-dependent formulation for the turbulent heat conductivity and a convective adjustment procedure. As a lake parameterisation scheme, that lake model was coupled to a number of atmospheric models. It enjoyed wide popularity...
in climate studies. MacKay (2005) developed a hybrid model, where the solution of the non-steady heat transfer equation on a numerical grid is combined with the bulk treatment of the upper mixed layer. Mironov (2008) developed a two-layer bulk model. The model, termed FLake, is based on a parametric representation of the evolving temperature profile in the upper mixed layer and in the lake thermocline and on the integral energy budgets for these layers.

In the present paper, the application of the lake model FLake to account for the effect of lakes in NWP is discussed. A brief description of FLake is given (see the next section and the Appendix), the implementation of FLake into the limited-area NWP model COSMO is discussed, and some results from a numerical experiment with the coupled COSMO-FLake NWP system, including the complete COSMO data assimilation cycle operational at the German Weather Service (DWD), are presented. Then, conclusions from the study are presented, and some challenging issues are outlined. It should be remarked that the primary goal of the present study is to demonstrate a satisfactory performance of FLake within COSMO with respect to the lake surface temperature and to the lake freeze-up and melting. Detailed verification of FLake performance in both stand-alone and coupled mode against observational data will be reported in other publications.

**Lake model FLake**

FLake is a bulk model capable of predicting the vertical temperature structure and mixing conditions in lakes of various depth on the time scales from a few hours to many years. The model is based on a two-layer parametric representation of the evolving temperature profile and on the integral budgets of energy for the layers in question. The structure of the stratified layer between the upper mixed layer and the basin bottom, the lake thermocline, is described using the concept of self-similarity (assumed shape) of the temperature-depth curve (Kitaigorodskii and Miropolsky 1970). The same concept is used to describe the temperature structure of the thermally active upper layer of bottom sediments and of the ice and snow cover. This approach, that relies on “verifiable empiricism” but still incorporates much of the essential physics, offers a very good compromise between physical realism and computational economy.

Using the integral approach, the problem of solving partial differential equations (in depth and time) for the temperature and turbulence characteristics is reduced to solving a number of ordinary differential equations for the time-dependent quantities that specify the evolving temperature profile. These are the mixed-layer temperature and the mixed-layer depth, the temperature at the water-bottom sediment interface, the mean temperature of the water column, the shape factor with respect to the temperature profile in the thermocline, the temperature at the upper surface of lake ice, and the ice thickness. Optionally, the bottom sediment module can be switched on to account for the interaction between the lake water and the bottom sediment. Then, two additional quantities are predicted, viz., the depth of the upper layer of bottom sediments penetrated by thermal wave and the temperature at that depth. Provision is made to explicitly account for the layer of snow above the lake ice. Then, prognostic equations are carried for the temperature at the snow upper surface and for the snow thickness. The snow module has not been comprehensively tested so far. The recommended choice at present is to account for snow above the lake ice implicitly (parametrically), namely, through an empirical temperature dependence of the ice surface albedo with respect to solar radiation.

FLake has been favourably tested against observational data through single-column numerical experiments. A detailed description of the model is given in Mironov (2008). A brief description of the model (of its configuration used in the present study) is given in the Appendix. Further information about FLake can be found at http://lakemodel.net.

**Implementation of FLake into the limited-area NWP model COSMO**

As a lake parameterisation scheme, FLake is implemented into the limited-area NWP model
COSMO (formerly called LM, Steppeler et al. 2003). In order to be incorporated into COSMO, or into any other NWP or climate model, FLake requires a number of two-dimensional external-parameter fields. These are, first of all, the fields of lake fraction (area fraction of a given numerical-model grid box covered by the lake water) and of lake depth. A lake-fraction field is developed on the basis of the Global Land Cover Characterization data set with 30 arc sec resolution. Since no tile approach is used in COSMO, i.e. each COSMO-model grid box is characterised by a single land-cover type, only the grid boxes with the lake fraction in excess of 0.5 are treated as lakes. A data set containing mean depths of a number of European lakes and of a few lakes from other regions of the world is developed at DWD. The lake-depth external-parameter field is generated using that data set and the lake-fraction field. It would be advantageous to specify some other characteristics that vary from one lake to another, e.g. the attenuation coefficient with respect to solar radiation. For most lakes, the information about the optical properties of water is not readily available. Then, a constant (default) value is utilised. Each lake is characterised by its mean depth. Results from single-column numerical experiments suggest that the use of a mean depth is the best choice as far as the prediction of the water surface temperature and of the ice characteristics is concerned. These quantities are of prime importance in NWP and climate modelling. This choice is in fact consistent with the one-dimensional character of the lake model. For lack of better data, it is the only choice for most small-to-medium size lakes. Deep lakes are currently treated with the “false bottom”. That is, an artificial lake bottom is set at a depth of 50 m. The use of such an expedient is justified since, strictly speaking, FLake is not suitable for deep lakes (because of the assumption that the thermocline extends down to the lake bottom). However, as the deep abyssal zones typically experience no appreciable temperature changes, using the false bottom one can expect FLake to produce satisfactory results.

In the present configuration, the bottom sediment module of FLake is switched off and the heat flux at the water–bottom sediment interface is set to zero. Although the snowfall rate is provided by COSMO, snow over lake ice is not considered explicitly. The effect of snow is accounted for implicitly through changes in the surface albedo with respect to solar radiation. A one-band exponential approximation of the decay law for the flux of solar radiation with the default value of the attenuation coefficient of 3 m$^{-1}$ is used. With this value, 95% of solar radiation is absorbed within the uppermost metre of the water column, yet the equilibrium depth of convectively mixed layer limited by the volumetric solar heating is not too small (see Eqs. 19 and 32 in the Appendix; convection affected by the volumetric solar heating is discussed in Mironov 2008). The surface fluxes of momentum and of sensible and latent heat are computed with the operational COSMO-model surface-layer scheme (Raschendorfer 1999, 2001).

### Numerical experiment with the coupled COSMO-FLake system

A numerical experiment, hereafter referred to as COFLEX, with the COSMO-FLake coupled system including the entire COSMO data assimilation cycle operational at DWD, is performed. The main goal with COFLEX was to see if the COSMO-FLake shows a reasonable performance with respect to the lake surface temperature (equal to the water temperature in the mixed layer or to the ice surface temperature if a lake is frozen), and to the lake freeze-up and melting. In order to save computational resources, the so-called LM1 numerical domain is used in COFLEX. That domain was operational at DWD until October 2005, prior to the operational implementation of the COSMO-EU whose numerical domain is much larger than the LM1 domain but the horizontal grid size is the same (ca. 7 km). The reader is referred to the COSMO web page, http://www.cosmo-model.org, for details of the COSMO model and its operational implementation at different NWP centres. Using the lake-fraction and the lake-depth external-parameter fields for the LM1 domain, the COSMO-FLake is run over a year, from 1 January to 31 December 2006. COFLEX is started on 1 January 2006, using the lake sur-
face temperature from the COSMO SST analysis as the initial condition and assuming no ice. The bottom temperature is set to the temperature of maximum density of fresh water, and the shape factor with respect to the temperature profile in the thermocline is set to its minimum value (see the Appendix and Mironov 2008, for details of the temperature profile parameterisation used in FLake). The initial mixed-layer thickness is set to 10 m or to the lake depth, whichever is smaller, i.e. mixing down to the lake bottom is assumed for the grid boxes with the lake depth smaller than 10 m. Once a cold start is made, the COSMO-FLake runs freely, i.e. without any correction of the lake surface temperature and of the other FLake variables.

Analysis of the COFLEX results indicates a satisfactory performance of FLake within COSMO. Many lakes present in the LMI domain are frozen and the ice melts in a reasonable time span. A detailed quantitative assessment of the simulated ice characteristics is difficult to make. A rough qualitative assessment is, however, possible, e.g. using World Lakes Database, http://www.ilec.or.jp/database/database.html, where data on the lake water temperature and on the duration of ice cover are given for a number of lakes (in many instances the information is available only for some particular years). Examination of results from COFLEX suggests that the lakes that freeze up in reality do also freeze up in the numerical experiment.

Reliable observational information on the duration of ice cover is available for Lake Balaton, Hungary (M. Vörös pers. comm.). Observations indicate that the lake was frozen on 10 January 2006 and the ice cover vanished in the beginning of March 2006. In COFLEX, the lake freeze-up occurs somewhat too late. This result is not surprising, however. The point is that the water temperature from the routine COSMO SST analysis used to initialise COSMO-FLake on 1 January 2006 was probably too high. As a result, it took a long time for COSMO-FLake to cool the lake water down to the freezing point. The time of ice break-up is predicted very well (Fig. 1). Unfortunately, no data on the ice thickness are available, making it impossible to comprehensively verify the COSMO-FLake performance with respect to the ice thickness.

For three lakes, viz., Lake Hjälmaren, Sweden, Lake Balaton, Hungary, and Lough Neagh, UK, the lake surface temperature predicted by COSMO-FLake (00 UTC values from the assimilation cycle), is compared with the lake surface temperature from the operational COSMO SST analysis (performed once a day at 00 UTC). Notice that in COFLEX the temperature from the routine COSMO SST analysis has no direct effect on the lake surface temperature that is now predicted by FLake. A substantial difference between the two temperatures is clearly seen (Figs. 2–4).

For lakes Hjälmaren and Balaton (Figs. 2 and 3), the difference is particularly striking during winter when it often exceeds 10 K. Such a large temperature difference is brought about by the procedure (outlined in the “Introduction”) used to determine the water surface temperature within the framework of the routine COSMO SST analysis. Recall that when the observational data for a water-type grid box are not available, the water surface temperature for that grid box is determined by means of interpolation between the nearest water-type boxes for which the surface temperature is known. Notice also that the COSMO SST analysis does not explicitly account for sea/lake ice (no ice model is used). For many lakes the interpolation is likely to occur between the sea points. During winter, the interpolation procedure yields a too high lake
surface temperature that stays well above the freezing point even though the lake considered is frozen in reality. As indicated by empirical data, both Lake Hjälmarne and Lake Balaton are frozen up for an appreciable length of time. The lakes are also frozen in COFLEX (see Fig. 1), whereas the surface temperature from the routine COSMO SST analysis indicates that both lakes remain ice free. This proves to be the case for many other lakes in the model domain (not shown). The situation is not encountered if FLake is used to predict the lake characteristics. Since FLake allows the lake surface temperature to adequately respond to atmospheric forcing, lakes freeze up and the lake surface temperature, which is then equal to the ice surface temperature, drops in response to surface cooling. An overestimation of the water (ice) surface temperature in the routine COSMO SST analysis may have far-reaching implications for the forecast quality. It leads to drastically increased surface fluxes of sensible and latent heat, particularly during winter. This in turn may result in the development of artificial cyclones over water bodies, leading to a considerable degradation of the COSMO performance. The situation will not occur if FLake is used to predict the lake surface temperature.

For Lough Neagh (Fig. 4), the difference between the two temperatures proves to peak during summer. Again, the temperature from the COSMO SST analysis is very likely a result of interpolation between the sea points. As the sea thermal inertia is high, the temperature from the SST analysis is an underestimation of the actual lake surface temperature during summer. A considerably higher temperature is predicted by FLake as it allows the lakes to adequately respond to the summer heating.

Conclusions and outlook

The lake model FLake is incorporated as a lake parameterisation scheme into the limited-area NWP model COSMO. Results from a numerical experiment with the coupled COSMO-FLake system, including the complete COSMO data assimilation cycle, indicate a good performance with respect to the lake surface temperature and to the freeze-up of lakes and the ice break-up. In particular, the use of FLake removes a significant overestimation of the lake surface temperature during winter that is typical of the routine COSMO SST analysis. Work is underway at
DWD to comprehensively assess the effect of FLake on the quality of the COSMO forecasts. Subject to successful verification of results from numerical experiments, FLake is expected to come into operational use.

Further development of FLake in terms of the model physics should address an extension of the temperature profile parameterisation to include the abyssal layer below the lake seasonal thermocline and the explicit treatment of snow over the lake ice. What is even more important from the application standpoint is the extension of the lake-depth data set in order to eventually cover the entire globe (see Kourzeneva 2010). This should allow the application of FLake over an arbitrary numerical domain. Apart from the lake depth, it is desirable to collect data on other lake characteristics, first of all, on the optical properties of the lake water.

One more challenging issue is the lake temperature spin-up following a cold start of FLake in an NWP or climate model. Lakes have a long memory. That is, erroneous initial conditions result in an erroneous heat content of the lake in question, leading to erroneous predictions of the lake surface temperature until the memory is faded. For stratified lakes this may last several months. Observations usually offer data on the water-surface temperature only, whereas the information about the entire temperature profile that is required for the FLake initialisation is lacking. A way out is to generate the so-called perpetual year solutions for the lakes present in a given numerical domain, using climatological mean meteorological data to specify atmospheric forcing. A perpetual year solution is obtained by repeating a year-long simulation, using one and the same annual cycle of forcing, until a periodic “perpetual year” state is achieved. That is, running the model for one more year will not change the annual cycle of the lake-model variables. Although a perpetual year solution represents a climatological mean state of the lake in question, not the state of that lake at a particular date, it is a reasonable zero-order approximation that should considerably reduce the lake-temperature spin-up time. As the perpetual year solutions are generated through the stand-alone single-column FLake runs, the procedure is computationally inexpensive.

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**Appendix. A brief description of the lake model FLake.**

In this section, a brief description of the lake model FLake is presented. A detailed description of the model is given in Mironov (2008), where an overview of previous studies, an extensive discussion of various parameterisation assumptions and of the model disposable constants and parameters, and references to relevant publications can be found. Here, only a short summary of FLake is presented, more specifically, of its simplified configuration implemented into COSMO. Recall that in COSMO-FLake, the heat flux through the water-sediment interface is set to zero so that the bottom sediment module is not used, and snow over lake ice is not considered explicitly.

The following quadratic equation of state of the fresh water is utilised:

\[
\rho_w = \rho_i \left[1 - \frac{1}{2} a_T (\theta - \theta_i)^2\right],
\]

(1)

where \(\rho_w\) is the water density, \(\rho_i = 999.98 \approx 1.0 \times 10^3\) kg m\(^{-3}\) is the maximum density of the fresh water at the temperature \(\theta_i = 277.13\) K, and \(a_T = 1.6509 \times 10^{-3}\) K\(^{-2}\) is an empirical coefficient. According to Eq. 1, the thermal expansion coefficient \(\alpha_T\) and the buoyancy parameter \(\beta\) depend on the water temperature \(\theta\):

\[
\beta(\theta) = g \alpha_T(\theta) = g a_T(\theta - \theta_i),
\]

(2)

where \(g = 9.81\) m s\(^{-2}\) is the acceleration due to gravity.

The following two-layer parametric representation of the evolving temperature profile is adopted (Kitaigorodskii and Miropolsky 1970):

\[
\theta = \begin{cases} 
\theta_s & \text{at } 0 \leq z \leq h \\
\theta_b - (\theta_b - \theta_s) \Phi(\zeta) & \text{at } h \leq z \leq D.
\end{cases}
\]

(3)

Here, \(\theta_s(t)\) is the temperature of the upper mixed layer of depth \(h(t)\), \(\theta_b(t)\) is the bottom temperature (temperature at the water-bottom sediment interface \(z = D\)), and \(\Phi(\zeta) \equiv [\theta (t) - \theta(z, t)]/[\theta_b(t) - \theta_s(t)]\) is a dimensionless “universal” function of dimensionless depth \(\zeta \equiv [z - h(t)]/[D - h(t)]\) that satisfies
the boundary conditions $\Phi_\theta(0) = 0$ and $\Phi_\theta(1) = 1$. With rare exception, the arguments of variables dependent on time $t$ and vertical co-ordinate $z$ (positive downward) are not indicated in what follows.

According to Eq. 3, $h, D, \theta_s, \theta_b$ and the mean temperature of the water column,

$$\bar{\theta} \equiv D^{-1} \int_0^D \theta dz,$$  \hspace{1cm} (4)

are related through

$$\bar{\theta} = \theta_s - C_\theta \left(1 - \frac{h}{D}\right)(\theta_b - \theta_s),$$ \hspace{1cm} (5)

where $C_\theta$ is the shape factor with respect to the temperature profile in the thermocline,

$$C_\theta \equiv \int_0^1 \Phi_\theta(\zeta) d\zeta.$$ \hspace{1cm} (6)

It should be emphasised at once that the exact form of the shape function is not required within the framework of the integral approach used to develop FLake. It is not $\Phi_\theta$ but the shape factor $C_\theta$ that enters the model equations.

The equation for the mean temperature of the water column (i.e. the equation of the total heat budget obtained by integrating one-dimensional heat transfer equation over $z$ from 0 to $D$) reads

$$D \frac{d\bar{\theta}}{dt} = \frac{1}{\rho_c c_w} \left[ Q_s + I_s - Q_b - I(h(D)) \right],$$ \hspace{1cm} (7)

where $c_w = 4.2 \times 10^3$ J kg$^{-1}$ K$^{-1}$ is the specific heat of water, $Q_s$ and $I_s$ are the values of the vertical heat flux $Q$ and of the heat flux due to solar radiation $I$, respectively, at the lake surface, and $Q_b$ is the heat flux through the lake bottom. The radiation heat flux $I_s$ that penetrates into the water is the surface value of the incident solar radiation flux from the atmosphere multiplied by $1 - \alpha_w$, where $\alpha_w$ is the albedo of the water surface with respect to solar radiation. The surface flux $Q_s$ is a sum of the sensible and latent heat fluxes and the net heat flux due to long-wave radiation at the air–water interface.

The equation of heat budget of the mixed layer reads

$$h \frac{d\theta_b}{dt} = \frac{1}{\rho_c c_w} \left[ Q_s + I_s - Q_h - I(h) \right],$$ \hspace{1cm} (8)

where $Q_h$ is the heat flux at the bottom of the mixed layer.

In the case of the mixed-layer stationary state or retreat, $dh/dt \leq 0$, the bottom temperature is assumed to remain constant:

$$\frac{d\theta_b}{dt} = 0.$$ \hspace{1cm} (9)

In the case of the mixed-layer deepening, $dh/dt > 0$, the following equation is used:

$$\frac{1}{2}(D - h)^2 \frac{d\theta_b}{dt} - \frac{d}{dt} \left[ C_{\theta_b} (D - h)^2 (\theta_s - \theta_b) \right] = \frac{1}{\rho_c c_w} \left[ C_{\theta_b} (D - h)(Q_h - Q_b) + (D - h)I(h) - \int_0^h \int_0^\zeta I(z) dz \right],$$ \hspace{1cm} (10)

where

$$C_{\theta_b} \equiv \int_0^1 \Phi_\theta(\zeta') d\zeta'.$$ \hspace{1cm} (11)

is a dimensionless parameter, and

$$C_{\theta} = \frac{2C_{\theta_b}}{C_\theta},$$ \hspace{1cm} (12)

is the shape factor with respect to the heat flux.
If \( h = D \), then both \( \theta_s \) and \( \theta_b \) are equal to the mean temperature of the water column that is computed from Eq. 7. Recall that the bottom heat flux \( Q_b \) is set to zero in the present model configuration. During convective mixed-layer deepening, \( h \) is determined from the following entrainment equation:

\[
\frac{Q_b}{Q_*} + \frac{C_{c1}}{w_*} \frac{dh}{dt} = C_{c1}.
\]

(13)

Here, \( Q_* \) and \( w_* \) are generalised convective scales of heat flux and of velocity, respectively, that account for the volumetric character of the solar radiation heating:

\[
Q_* = Q_s + I_s + I(h) - \frac{2}{h_s^2} \int_0^{h_s} I(z) dz, \quad w_* = \left[ -h \beta(\theta_s) Q_s / \rho_o c_w \right]^{1/3}.
\]

(14)

and \( C_{c1} = 0.17 \) and \( C_{c2} = 1.0 \) are dimensionless constants.

The depth of a stably or neutrally stratified wind-mixed layer is determined from the following relaxation-type rate equation:

\[
\frac{dh}{dt} = -\frac{h - h}{t_{rh}}.
\]

(15)

Here, \( t_{rh} \) is the relaxation time scale estimated as

\[
t_{rh} = \frac{h}{C_{rh} u_*},
\]

(16)

where \( u_* = |\tau_z / \rho_o|^{1/2} \) is the surface friction velocity, \( \tau_z \) being the surface stress, and \( C_{rh} = 0.03 \) is a dimensionless constant. The equilibrium mixed-layer depth \( h_e \) is computed from

\[
\left( \frac{f h_s}{C_{rh} u_*} \right)^2 + \frac{h_s}{C_{rh} L} + \frac{h_s N}{C_{rh} c_w} = 1,
\]

(17)

where \( f = 2 \Omega \sin \varphi \) is the Coriolis parameter, \( \Omega = 7.29 \times 10^{-5} \) \( \text{s}^{-1} \) is the angular velocity of the earth’s rotation, \( \varphi \) is the geographical latitude, \( L = u_* [ \beta(\theta_s) Q_s / \rho_o c_w ] \) is the Obukhov length, \( N \) is the buoyancy frequency below the mixed layer, and \( C_{rh} = 0.5 \), \( C_{rh} = 0.10 \), and \( C_{rh} = 20 \) are dimensionless constants. A generalised formulation of the Obukhov length that accounts for the vertically distributed character of the solar radiation heating is used. A mean-square buoyancy frequency in the thermocline,

\[
\overline{N^2} = \left( (D - h)^{-1} \int_0^D N^2 dz \right)^{1/2}.
\]

(18)

is used as an estimate of \( N \) in Eq. 17.

The equilibrium mixed-layer depth is limited from below by the depth of a convectively mixed layer whose deepening driven by the surface cooling \( (Q_s < 0) \) is arrested by the volumetric radiation heating \( (I > 0) \). The equilibrium depth \( h_c \) of such layer is computed from

\[
Q_s(h_c) = Q_s + I_s + I(h_c) - \frac{2}{h_c} \int_0^{h_s} I(z) dz = 0,
\]

(19)

where a finite solution \( h_c < \infty \) exists if \( -Q_s I_s < 1 \). If Eq. 17 predicts a very shallow stably-stratified equilibrium mixed layer to which the mixed layer (whose current depth \( h \) exceeds \( h_c \)) should relax, then it is required that \( h_c \geq h \). This limitation is imposed if the mixed-layer temperature \( \theta_s \) exceeds the temperature \( \theta_t \) of maximum density of fresh water and \( Q_s(h) > 0 \) [a negative \( Q_s(h) \) indicates that the mixed layer is convective, so that Eq. 13 should be used to compute \( h \)].
The approach used to describe the temperature structure of the lake ice is conceptually similar to the approach used to describe the temperature structure of the lake thermocline. The following parametric representation of the evolving temperature profile within the ice is adopted (cf. Eq. 3):

$$\theta = \theta_i - (\theta_i - \theta_f) \Phi_i(\zeta_i) \quad \text{at} \quad -h_i \leq \zeta_i \leq 0.$$  

(20)

where \( z \) is the vertical co-ordinate (positive downward) with the origin at the ice-water interface, \( h_i(t) \) is the ice thickness, \( \theta_f = 273.15 \text{ K} \) is the fresh-water freezing point, and \( \theta_i(t) \) is the temperature at the ice upper surface. Dimensionless “universal” function \( \Phi \equiv \frac{[\theta_i - \theta(z, t)] / [\theta_i - \theta_f]}{\zeta_i} \) of dimensionless depth \( \zeta_i = -z/h_i(t) \) satisfies the boundary conditions \( \Phi_i(0) = 0 \) and \( \Phi_i(1) = 1 \).

The equation of the heat budget of the ice layer reads

$$\frac{d}{dt} \left[ \rho_i c_i \theta_i \left[ \left( \theta_i - \theta_f \right) \Phi_i(\zeta_i) \right] \right] - \rho_i c_i \theta_i \frac{d h_i}{dt} = Q_s + I_s - I(0) + \kappa_i \frac{\theta_i - \theta_f}{h_i} \Phi_i(0),$$  

(21)

where \( \rho_i = 9.1 \times 10^2 \text{ kg m}^{-3}, c_i = 2.1 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} \) and \( \kappa_i = 2.29 \text{ J m}^{-1} \text{ s}^{-1} \text{ K}^{-1} \) are the density, the specific heat and the heat conductivity of ice, respectively, \( Q_s \) and \( I_s \) are the values of \( Q \) and \( I, \) respectively, at the ice upper surface \( z = -h_i(t), \) and \( \Phi_i(0) = d\Phi_i/d\zeta_i \) at \( \zeta_i = 0. \) The radiation heat flux \( I_s \) that penetrates into the ice interior is the surface value of the incident solar radiation flux from the atmosphere multiplied by \( 1 - \alpha_i, \) where \( \alpha_i \) is the ice surface albedo with respect to solar radiation. The dimensionless parameter \( C_i \) is the shape factor with respect to the temperature profile within the ice:

$$C_i \equiv \int_0^1 \Phi_i(\zeta_i) d\zeta_i.$$  

(22)

Equation 21 serves to determine \( \theta_i \) when this temperature is below the freezing point \( \theta_f, \) i.e. when no melting at the ice upper surface takes place. During the ice melting from above, \( \theta_i \) remains equal to \( \theta_f. \)

During the ice growth or ice melting from below (these occur as \( \theta_i < \theta_f \)), the ice thickness is computed from the following equation:

$$L_i \frac{d \rho_i h_i}{dt} = Q_s + \kappa_i \frac{\theta_i - \theta_f}{h_i} \Phi_i(0),$$  

(23)

where \( L_i = 3.3 \times 10^5 \text{ J kg}^{-1} \) is the latent heat of fusion, and \( Q_s \) is the heat flux in the near-surface water layer just beneath the ice. If the right-hand side of Eq. 23 is negative (this may occur due to a negative \( Q_s \)), ice ablation takes place.

During the ice melting from above, the following equation is used:

$$L_i \frac{d \rho_i h_i}{dt} = -(Q_s + I_s) + Q_s + I(0),$$  

(24)

that holds as the atmosphere heats the ice upper surface and \( \theta_i \) is equal to \( \theta_f. \)

The evolution of the temperature profile beneath the ice is described as follows. The temperature at the ice-water interface is fixed at the freezing point, \( \theta = \theta_f. \) The mean temperature of the water column is computed from Eq. 7, where \( Q_s \) and \( I_s \) are replaced with \( Q_w \) and \( I(0), \) respectively. If the bottom temperature is less than the temperature of maximum density, \( \theta_b < \theta_r, \) the mixed-layer depth and the shape factor with respect to the temperature profile in the thermocline are kept unchanged, \( dh/dt = 0 \) and \( dC_{\theta}/dt = 0, \) and the bottom temperature is computed from Eq. 5. If the entire water column appears to be mixed at the moment of freezing, i.e. \( h = D \) and \( \theta_s = \theta_b, \) the mixed layer depth is reset to zero, \( h = 0, \) and the shape factor is reset to its minimum value, \( C_{\theta} = C_{\theta}^{\text{min}}. \) As the bottom temperature reaches the temperature of maximum density, its further increase is prevented and \( \theta_b \) is kept constant equal to \( \theta_r. \) If \( h > 0, \) the shape factor \( C_{\theta} \) is kept unchanged, and the mixed-layer depth is computed from Eq. 5. As the mixed-layer depth approaches zero, Eq. 5 is used to compute the shape
factor \( C_\theta \) that in this regime increases towards its maximum value, \( C_\theta = C_\theta^{\text{max}} \) (estimates of \( C_\theta^{\text{min}} \) and \( C_\theta^{\text{max}} \) are given below). If \( h = 0 \), the heat flux from water to ice is estimated from

\[
Q_v = -k_w \frac{\theta_i - \theta_r}{D} \max\left[1, \Phi_\theta'(0)\right],
\]

(25)

where \( k_w = 5.46 \times 10^{-1} \text{ J m}^{-1} \text{ s}^{-1} \text{ K}^{-1} \) is the molecular heat conductivity of water, and \( \Phi_\theta'(0) = d\Phi_\theta/d\xi \) at \( \xi = 0 \). If \( h > 0 \), \( Q_v = 0 \).

The shape factor with respect to the temperature profile in the thermocline is computed from

\[
dC_\theta = \text{sign} \left( \frac{dh}{dt} \right) \frac{C_\theta^{\text{max}} - C_\theta^{\text{min}}}{t_c}, \quad C_\theta^{\text{min}} \leq C_\theta \leq C_\theta^{\text{max}},
\]

(26)

where \( \text{sign} \) is the sign function \( \text{sign}(x) = -1 \) if \( x \leq 0 \) and \( \text{sign}(x) = 1 \) if \( x > 0 \), and \( C_\theta^{\text{min}} = 0.5 \) and \( C_\theta^{\text{max}} = 0.8 \) are minimum and maximum values of the shape factor, respectively. The shape factor \( C_\theta \) evolves towards its maximum value during the mixed-layer deepening, and towards its minimum value during the mixed-layer stationary state or retreat. The adjustment occurs on a relaxation time scale \( t_c \) estimated as

\[
t_c = \frac{(D-h)^2}{C_n u_i^2}, \quad u_i = \max(w_i, u_i),
\]

(27)

where \( C_n = 0.003 \) is a dimensionless constant, and the mean-square buoyancy frequency in the thermocline is given by Eq. 18. Notice that Eqs. 26 and 27 are used during the period of open water. During the period of ice cover, a different procedure is used as outlined above.

The dimensionless parameter \( C_{\theta\theta} \) defined through Eq. 11 is given by

\[
C_{\theta\theta} = \frac{11}{18} C_\theta - \frac{7}{45},
\]

(28)

and the quantity \( \Phi_\theta'(0) \) that enters Eq. 25 is given by

\[
\Phi_\theta'(0) = \frac{40}{3} C_\theta - \frac{20}{3}.
\]

(29)

The shape factor with respect to the temperature profile within the ice is computed from

\[
C_i = \frac{1}{2} - \frac{1}{12} (1 + \Phi_s) \frac{h_i}{h_i^{\text{max}}},
\]

(30)

where \( h_i^{\text{max}} = 3 \text{ m} \) and \( \Phi_s = 2 \). The quantity \( \Phi_i'(0) \) that enters Eqs. 21 and 23 is given by

\[
\Phi_i'(0) = 1 - \frac{h_i}{h_i^{\text{max}}}. \tag{31}
\]

The exponential approximation of the decay law for the flux of solar radiation is used:

\[
I = I_0 \sum_{k=1}^n a_k \exp\left[-\gamma_k (z + h_i)\right],
\]

(32)

where \( I_0 \) is the surface value of the incident solar radiation heat flux multiplied by \( 1 - \alpha \), \( \alpha \) being the albedo of the water surface or of the ice surface with respect to solar radiation, \( n \) is the number of wavelength bands, \( a_k \) are fractions of the total radiation flux for different wavelength bands, and \( \gamma_k \) are attenuation coefficients for different bands. The attenuation coefficients are piece-wise constant functions of \( z \), i.e. they have different values for water and ice but remain constant within these media.

The following parameterization of the ice surface albedo with respect to solar radiation is adopted:

\[
\alpha_i = \alpha_i^{\text{max}} - \alpha_i^{\text{min}} \exp\left[-C_\theta (\theta_i - \theta) / \theta_i\right]. \tag{33}
\]
where $\alpha_i^{\text{max}} = 0.6$ and $\alpha_i^{\text{min}} = 0.1$ are maximum and minimum values of the ice albedo, respectively, and $C_\alpha = 95.6$ is a fitting coefficient. Equation 33 is meant to implicitly account, in an approximate manner, for the presence of snow over lake ice and for the seasonal changes of $\alpha_i$. During the melting season, the ice surface temperature is close to the fresh-water freezing point. The presence of wet snow, puddles, melt-water ponds and leads on the surface of melting ice results in a decrease of the area-averaged surface albedo. The water surface albedo with respect to solar radiation, $\alpha_w$, is taken to be constant equal to 0.07.

The FORTRAN code of the lake model FLake is freely available from http://lakemodel.net, where further information about the model can also be found.